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Laplacian Eigenmaps Manifold Learning and Anomaly Detection Methods for Spectral Images

by

Marcela Munoz Reales

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
of Master of Science
in Applied and Computational Mathematics

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Abstract

Laplacian Eigenmaps Manifold Learning and Anomaly Detection Methods for Spectral Images

Marcela Munoz Reales

Supervising Professor: Dr. William Basener

Spectral images provide a large amount of spectral information about a scene, but sometimes when studying images, we are interested in specific components. It is a difficult problem to separate the relevant information or what we call interesting from the background of a spectral image, even more so if our target objects are unknown. Anomaly detection is a process by which algorithms are designed to separate the anomalous (different) points from the background of an image. The data is complex and lives in a high dimension, manifold learning algorithms are used to analyze data that lives in a high dimensional space, but that can be represented as a lower dimensional manifold embedded in the high dimensional space. Laplacian Eigenmaps is a manifold learning algorithm that applies spectral graph theory methods to perform a non-linear dimensionality reduction that preserves local neighborhood information. We present an

approach to reduce the dimension of the data and separate anomalous pixels in spectral images using Laplacian Eigenmaps.

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Chapter 1

Introduction

Spectral images are digital images that contain measurements of wavelengths of light so that a spectrum is provided for each pixel instead of the usual red, green and blue. They contain complex high dimensional data that is difficult to study in its original form. Imaging processing methods are used in the study of the information provided by spectral images. The three most important uses of image processing are clustering and classification, anomaly detection, and target detection. We are interested in the problem of anomaly detection.

There are several mathematical tools that can be used to extract information from spectral data. Statistical models using principal component analysis (PCA) and Reed-Xiaoli anomaly detector can be applied to analyze the background information and create a ranking of anomalous pixels in spectral images. PCA is a dimensionality reduction algorithm that finds a lower dimension representation

by embedding the data into a linear subspace. RX (Reed-Xiaoli anomaly detector) is the most popular anomaly detection algorithm, it uses the covariance matrix and the distance to the mean to locate anomalies. Spectral graph theory methods can be used to study spectra variations, and detect anomalies; images can be modeled as sets of connected components where pixels are vertices with edges connecting them under specific conditions. TAD (Topological Anomaly Detection algorithm) creates a graph using the pixel's spectra as vertices that are connected if they are spectrally similar.

Manifold learning algorithms are used to analyze data that lives in a high dimensional space, but that can be represented as a lower dimensional manifold embedded in the high dimensional space. Laplacian Eigenmaps is a manifold learning algorithm that uses spectral graph theory concepts to represent spectral data as a graph, using the pixels' spectra as vertices that are connected if their spectra are similar; one can construct a Laplacian matrix from the degree of the vertices and perform an eigen-decomposition to aid in the search for pixels that have anomalous spectra.

We present an approach that uses a Laplacian Eigenmaps algorithm for anomaly detection in spectral images.

Chapter 2

Imaging Processing

Remote sensing tools were designed to capture information about objects without coming into direct contact with them. Remote sensing instruments can be used in many applications such as to help in the study of crops, characterization of soils, and mineral exploration [1].

Multi-spectral images are a type of image used in remote sensing. They can provide information undetectable by the human eye, capturing images in four or more wavelengths of light, and stored in a file with one band for each wavelength. As remote sensing progressed, hyperspectral imagery was introduced. Hyperspectral images have many narrow bands that provide data from across the electromagnetic spectrum. Each pixel of the image contains many spectral bands that allow material identification.

Materials have a reflectance spectrum that characterizes them; this is called the spectral signature. In an ideal world the spectral signature of materials

would remain unchanged under changing circumstances. But in reality, the reflectance spectrum of most materials exhibits variability caused by errors in the sensor, atmospheric and environmental changes, and variation in the amount of light absorbed or reflected by the material [2]. It is also worth noting that man-made material show less spectral variability than objects of the natural environment such as grass, soil, etc.

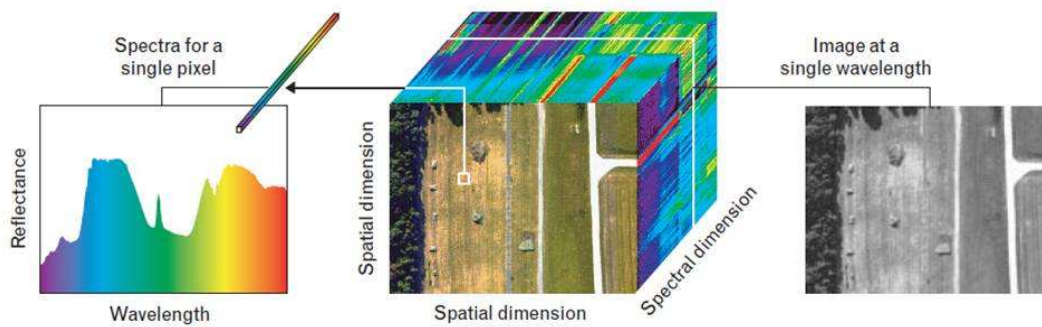


Figure 2.1: Data cube structure. The figure shows the spectral cube for an image (middle), a view as a set of spectra per each pixel (left), or as a single image for each single spectral channel (right)[1]

Spectral data generated by spectral imagery contains three-dimensional spatial-spatial-spectral measurements, which can be visualized with what is called the spectral cube [1]. The x and y (spatial) dimensions of the data cube for each pixel are the two-dimensional image that the human eye can see, the z dimension contains spectral information captured by the few hundred bands of the hyperspectral imaging sensor. Therefore the most important and dependable

information comes from the spectral data.

The three most important uses of image processing are:

unmixing/clustering/classification, anomaly detection, and target detection. Spectral unmixing and classification algorithms seek to separate each pixel's spectrum by identifying the endmember spectra for the image and their proportion in the pixel [3]. Anomaly detection aims to separate the anomalous points from the background of an image. Target detection is similar to anomaly detection but with the difference that the objects of interest have known characteristics. Two desirable characteristics of target and anomaly detection algorithms, other than being computationally efficient, are high probability of detection and, low probability of false alarm (low false-alarm-rate).

The first approach of many imaging process algorithms is dimension reduction. The objective of dimension reduction is to represent the signal in a minimal way that saves the necessary information to perform a successful unmixing process in a lower dimensional space [3].

2.1 Classification

Classification is the process of identifying the largest components of the image, and organizing the pixels according to the endmember component they belong to.

The spectrum of a mixed pixel contains a mixture of materials, either as a result of low spatial resolution, or a pixel that is composed of a homogeneous mixture of materials. Spectral unmixing yields the endmembers and the proportion of each in the pixel, this can be used for clustering and classification. Endmembers are natural or man-made materials that are part of the image, for example, grass, water, or different types of concrete [3]. The largest endmember components of the image can be classified, since they are part of the majority of pixels, and interpretation of the scene can be done by analyzing the clusters they form.

2.2 Anomaly Detection

Anomaly detection is the process of identifying pixels in an image whose spectra is very dissimilar from the spectra of the background of the image. Anomaly detection algorithms look for a small number of objects in a scene, for this reason, classification methods are not typically used, because most of the time the image provides little information about objects of interest or they are not clearly resolved [1]. In other words, large components of the image are only used in anomaly detection algorithms as a point of reference to identify anomalous pixels.

Anomaly Detection can be more effective when comparing a pixel in an image to its immediate vicinity. One of the most important uses is to recognize man-made structures or objects from natural surroundings, a car or a house in the middle of the forest. It can also be used to increase the probability of detection and area covered for search and rescue operations at sea [1].

2.3 Target Detection

Target detection is the process of identifying pixels in an image whose spectra matches a known target spectrum or nature. General applications try to identify small groups of objects with known shape or spectrum in an image. Target detection is widely used for agricultural applications to look for crop infestation. It can also be used in conjunction with anomaly detection, this is done, by extracting a set of materials that are anomalous or different, and then verify if the materials match a specific target [2].

Chapter 3

Graph Theory

Definitions from [4].

Definition 1. A simple *graph* G is a finite nonempty set of objects called *vertices* denoted by $V(G)$, together with a set of unordered pairs of distinct vertices of G called *edges* denoted by $E(G)$.

Definition 2. The *degree* of a vertex v in a graph G is the number of edges of G incident with v , denoted by d_v .

Definition 3. A *graph* G is connected if and only if there exists a path between every pair of vertices u and v in G . Otherwise the graph is disconnected.

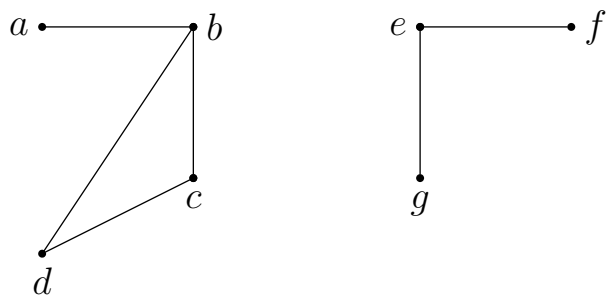
Definition 4. A *component* of a *graph* G is a subgraph induced by the vertices of G .

3.1 Example

Consider the following graph G define by vertex and edge sets

$$V(G) = \{a, b, c, d, e, f, g\} \quad \text{and} \quad E(G) = \{(a, b), (b, c), (b, d), (c, d), (e, f), (e, g)\}$$

G :



G is disconnected with 2 components.

We can list the degree values for the vertices of G in a degree table as follows,

Vertex	a	b	c	d	e	f	g
Degree	1	3	2	2	2	1	1

Chapter 4

Algorithms for Anomaly Detection

4.1 PCA

Principal Component Analysis (PCA) is a linear dimensionality reduction method; its approach is to embed high dimensional data into a linear subspace while preserving the most variance in the data possible. It does an orthogonal linear transformation in which the variance of the data is maximal. PCA provides a linear mapping onto the d principal eigenvectors of the covariance matrix, which is solved by the d principal eigenvalues λ . The low dimensional representation is obtained by mapping the eigenvalues λ onto the linear mapping [5].

4.2 RX

The *Reed-Xiaoli anomaly detector* commonly known as RX or RXD is a popular anomaly detection algorithm. It searches for objects in the minor eigenvalues. Using every pixel of the image, the mean μ and the covariance matrix Γ are computed, and the Mahalanobis distance from the mean to each pixel are used to detect anomalies. For a test pixel x using the RX algorithm we get:

$$RX(x) = (x - \mu)\Gamma^{-1}(x - \mu)$$

Which is equal to the number of standard deviations away from the mean of the data as a multivariate normal distribution [6]

Local RX anomaly detection algorithm compares anomalous pixels to their immediate vicinity's background rather than the entire image. This is achieved by a dual window with a smaller window within a larger outer one, and computing the mean and covariance using the pixels in the larger window, the pixels in the smaller one are not included in the computation [6].

In Envi, the ENVI RX Anomaly Detection Tool uses the RX algorithm to find anomalies in spectral images. It outputs a grayscale image where the anomalous

pixels are brighter than the background pixels.

4.3 TAD

Messinger, Basener and Ientilucci proposed an anomaly detection algorithm for spectral imagery called Topological Anomaly Detection (TAD). Their approach is to treat the spectral data in their k -dimensional space; without doing a dimension reduction. The topology of the data is analyzed and points are separated into background and anomalies. TAD uses *combinatorial topology* which refers to studying the structure of the non-parametric space where the objects of interest live using combinatorial methods.

The algorithm creates a graph using a subset of the image's pixels spectra with data points x_1, x_2, \dots, x_n from the spectral image as the vertices, adding an edge from x_i to x_j if pixel x_i is spectrally similar to x_j . A subset is used for computational efficiency. The large component of the graph is assumed to be the background, and smaller components that contain small percentages of the pixels in the image are ranked according to their distance to the background cluster, and are declared anomalies [6].

Compared to other statistical methods, TAD has the advantage that measurements are taken by calculating the distances between neighboring data points, instead of the distance to the mean of the total data, which aids in the successful detection of more anomalous points.

Chapter 5

Manifold Learning Algorithms

Complex data sets are hard to study in their original form. Manifold Learning algorithms were developed to analyze data that lives in a high dimensional space, with the belief that the data can be represented in a lower dimensional manifold of dimensionality d , embedded in a high dimensional space of dimensionality D , such that $d < D$ [7]

.

Definitions from [7]

Definition 5. A *homeomorphism* is a continuous function whose inverse is also a continuous function.

Definition 6. A *d -dimensional manifold* M is a set that is locally homeomorphic with \mathbb{R}^d . For each $x \in M$, there is an open neighborhood around x , N_x , and a homeomorphism $f : N_x \rightarrow \mathbb{R}^d$. The neighborhoods are denoted *coordinate patches*, and the map is denoted a *coordinate chart*. The image of the coordinate charts is called the *parameter space*.

Definition 7. A manifold is considered a *smooth (differentiable) manifold*, if each coordinate chart (map) is differentiable with a differentiable inverse.

An embedding of a manifold M into \mathbb{R}^d is a smooth (differentiable) homeomorphism from M into another space that is a subset of \mathbb{R}^d .

5.1 Isomap

Isometric feature mapping (Isomap) [7] is a well-known manifold learning algorithm. Its approach is to find the geodesic distances between neighboring data points using shortest-path distances. Then it uses the Multidimensional Scaling (MDS) method, which given a matrix of dissimilarity $D \in \mathbb{R}^{n \times n}$ constructs a set of points such that their Euclidean distances match the ones in D , to find points in a low-dimensional Euclidean space that match the nearest neighbors geodesic distances found in the first step.

Isomap is a good method to study large data sets, since it gives an estimate of the dimensionality of the underlying manifold.

5.2 Locally Linear Embedding

LLE [7] is another manifold learning algorithm that was introduced around the same time as Isomap. The scheme of LLE is to think about a manifold as a collection of coordinate patches that overlap. With a manifold that is sufficiently smooth, these patches, and the chart from the manifold to \mathbb{R}^d will be roughly

linear. By finding the linear patches and describing their geometry, one can find a mapping to \mathbb{R}^d that preserves their geometry and is almost linear.

5.3 Laplacian Eigenmaps

Laplacian Eigenmaps [7] is a manifold learning algorithm that makes use of spectral graph theory to represent the data as a graph, with nodes connected by edges if they are near or of similar nature. It uses an approximation to the manifold structure by the adjacency matrix computed from the data points and their distances in the manifold. A weighted Laplacian matrix is created from the adjacency matrix, with weights given by the heat kernel of the Laplace Beltrami operator in the heat equation. By doing an eigenvalue decomposition, one can obtain a vast amount of information about the underlying structure, including geometric characterization of the data.

Belkin and Niyogi showed that in some instances, the results obtained by the Laplacian Eigenmaps algorithm are equivalent to those obtained by the LLE algorithm [8].

Chapter 6

Images

The images we used were taken with a WorldView-2 satellite. WorldView-2 is the first high-resolution 8-band multispectral commercial satellite. It also contains a high-resolution panchromatic band. The first four primary bands are blue, green, red, and near-infrared bands. The additional bands are red edge for better accuracy on vegetation, coastal band for water color studies, yellow band, and an additional longer wavelength near infrared band. It operates at an altitude of 770 Km, with a 46 cm panchromatic resolution and 1.84 m multispectral resolution. Figure 6.1 shows the spectral responses of the bands [9].

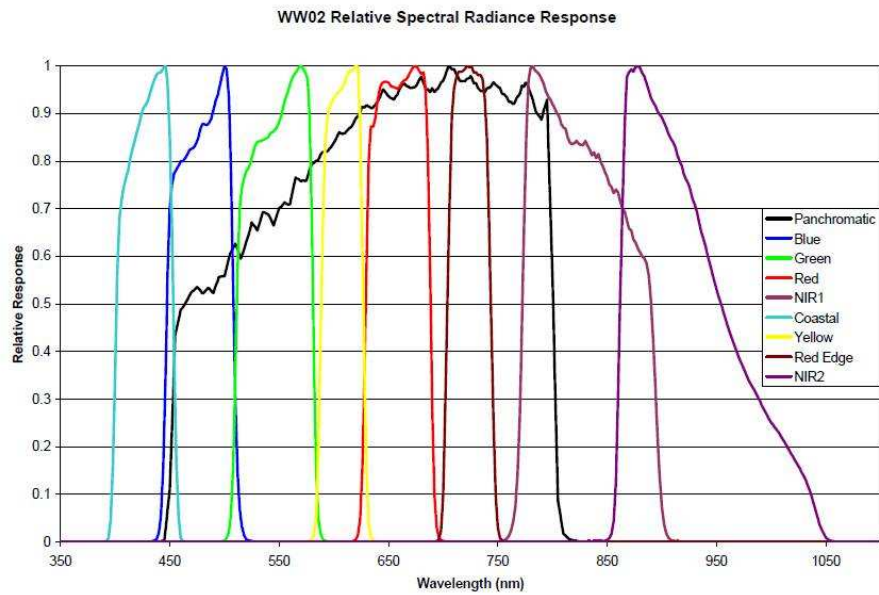


Figure 6.1: Spectral response of WorldView2 panchromatic and multispectral imager [9]

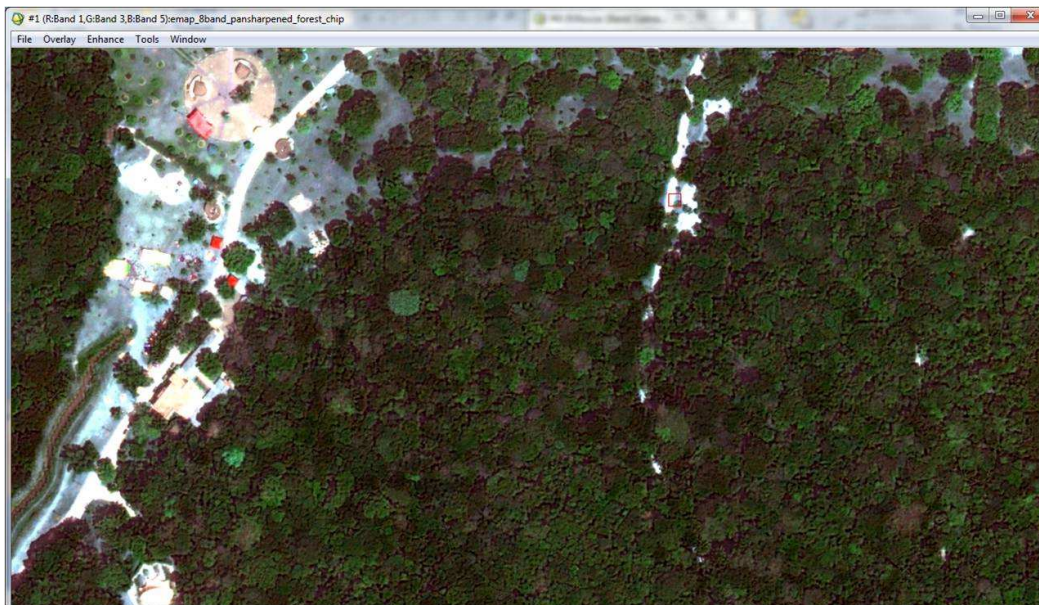


Figure 6.2: World View 2 Image

Chapter 7

Laplacian Eigenmaps

Laplacian Eigenmaps is a dimension reduction algorithm similar to *PCA*, but using graph theory methods instead of statistics.

A spectral image is composed of pixels with n spectral bands. We can take each pixel and treat it as a point in a n -dimensional space, where n is the number of bands. We believe that much of the data of interest lives in a lower dimension, and this the motivation to use *Laplacian Eigenmaps*.

In their 2002 paper Belkin and Niyogi proposed an approach to obtain and represent low dimensional data embedded in a high dimensional space. Their method uses the relationship of the graph Laplacian, the Laplace Beltrami operator on the manifold, and the connections to the heat equation [8]. The advantage of this algorithm is that it is computationally efficient and utilizes neighborhood information, which makes it a good candidate to assist in the problem of anomaly detection for spectral imaging, since we can exploit the fact that

pixel's spectrum is similar in the background, and hope that anomalous pixels would stick far out.

The algorithm applied to spectral imagery data computes a low-dimensional representation of the image data in which the distances between a pixel and its k nearest neighbors (in spectral space) are minimized. It is possible to construct a graph from an image by using pixels as vertices, and adding an edge between two pixels i and j if their spectra are similar, in such a way that there exist an edge (i, j) in the graph if the Euclidean distance from the spectrum of i to the spectrum of j is less than a defined threshold t .

We then create a Laplacian matrix from the degree of the vertices in the graph, and use the eigenvalues and corresponding eigenvectors of the Laplacian matrix to represent the image in a lower dimensional space. This can be used to search for anomalies in the network, since the spectrum of an anomalous pixel should be significantly different from that of its neighbors.

There is an idea derived from perturbation theory [10], that suggests that in the optimal case the first (smallest) eigenvectors of the Laplacian are indicator vectors, so that the entry is zero if the vertex is not in the group. In real-life applications, eigenvectors are more resistant to normal fluctuations in the data,

and reflect minimal changes that help to separate the graph into different components. This idea could be of assistance in the analyzes of results for Laplacian Eigenmaps.

7.1 Definitions

Definition 8. The *identity matrix* I on n vertices is defined by:

$$I(i, j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 9. The degree matrix D is a diagonal matrix with the (j, j) th entry having value d_j .

Definition 10. The *adjacency matrix* W for a given graph G is defined by:

$$W(i, j) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 11. The *Laplacian* matrix L for a simple graph G is defined by:

$$L = D - W$$

Definitions from [11]:

Definition 12. The *normalized Laplacian* of G is defined by the matrix

$$\mathcal{L}(i, j) = \begin{cases} 1 & \text{if } i = j \text{ and } d_v \neq 0, \\ \frac{-1}{\sqrt{d_i d_j}} & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

given by

$$\mathcal{L} = D^{-1/2} L D^{-1/2}$$

Which is equivalent to [10]

$$\mathcal{L} = I - D^{-1/2} W D^{-1/2}$$

The decision of using the normalized Laplacian is in part intuitive and in

part motivated by several sources such as ([11, 12]) that agree that through empirical studies the normalized Laplacian best capture the underlying graph's spectral geometry, and because it contains information about a random walk used in stochastic processes.

The use of the normalized Laplacian was intuitive as well because for highly connected nodes,

$$\mathcal{L}(u, v) = \frac{-1}{\sqrt{d_i d_j}}$$

is very small, whereas for poorly connected nodes it is considerably larger. Since the objective is to identify anomalous pixels whose spectra is very different from the rest of the pixels in the image, when doing the eigenvalue decomposition the results obtained for the anomalies that we want to detect stick out from the rest.

7.2 Properties of the Laplacian

The matrix L as defined in 7.1 has the following properties from [10]:

1. L is a real symmetric matrix

$$2. \forall f \in R^N f^T L f = \frac{1}{2} \sum_{i,j} W_{i,j} (f_i - f_j)^2$$

3. L is a positive-semidefinite matrix

4. All eigenvalues of L are positive and real. This results from property 3.

5. An eigenvalue that is equal to 0 indicates that the graph is connected. The number of connected components of the graph is equal to the number of eigenvalues that are equal to 0.

These properties are equivalent for \mathcal{L} ,

Property 1, \mathcal{L} is symmetric because L is symmetric. Multiplying both sides by the same diagonal matrix results in a symmetric matrix, therefore $D^{-1/2} L D^{-1/2}$ is symmetric.

Property 2 is as follows:

$$\forall f \in R^N f^T \mathcal{L} f = \frac{1}{2} \sum_{i,j} W_{i,j} \left(\frac{f_i}{\sqrt{d_i}} - \frac{f_j}{\sqrt{d_j}} \right)^2$$

Proof

$$\frac{1}{2} \sum_{i,j} W_{i,j} \left(\frac{f_i}{d_i} - \frac{f_j}{\sqrt{d_j}} \right)^2 = \frac{1}{2} \sum_{i,j} W_{i,j} \left(\frac{f_i^2}{d_i} - 2 \frac{f_i}{\sqrt{d_i}} \frac{f_j}{\sqrt{d_j}} + \frac{f_j^2}{d_j} \right) \quad (7.1)$$

$$= \sum_i f_i^2 - \sum_{i,j} W_{i,j} \left(\frac{f_i}{\sqrt{d_i}} \frac{f_j}{\sqrt{d_j}} \right) \quad (7.2)$$

$$= f^T f - f^T D^{-1/2} W D^{-1/2} f \quad (7.3)$$

$$= f^T \left(I - D^{-1/2} W D^{-1/2} \right) f \quad (7.4)$$

$$= f^T \mathcal{L} f \quad (7.5)$$

□

Property 3, \mathcal{L} is positive-semidefinite, such that $\forall x \in R^N x^T \mathcal{L} x \geq 0$. Follows from Property 2.

Property 4 follows from \mathcal{L} being positive-semidefinite. Refer to [10] for a proof of Property 5.

7.3 Algorithm

1. *Read in the image file*

Let X be a $n \times b$ array that contains the spectra of the n pixels in the image, where b is the number of bands in the image.

2. *Distance matrix*

Compute the Euclidean distances. Let S be the distance matrix of G where the i, j th entry corresponds to the Euclidean distance from the spectra of pixel i to the spectra of pixel j .

3. *Construct the graph*

Define a threshold t such that $t \in R^+$. Let G be graph with vertex set $V(X) = x_1, x_2, \dots, x_n$, together with edge set $E(X)$ and if i and $j \in V(X)$, then (i, j) is an edge in G if and only if $S(i, j) \leq t$.

4. *Adjacency matrix*

Create an adjacency matrix W from G .

5. *Compute the normalized Laplacian*

There is a nice way of calculating the *Laplacian* $\mathcal{L}(u, v)$ of a simple graph

G [11], first consider the matrix L of the form,

$$L(i, j) = \begin{cases} d_j & \text{if } i = j, \\ -1 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Compute the degrees of the vertices from W .

The Laplacian \mathcal{L} is of the form defined by 7.1. It can be computed as follows:

$$\mathcal{L} = D^{-1/2} L D^{-1/2}$$

6. Spectrum of the graph

Perform an eigen-decomposition on \mathcal{L} , to obtain eigenvalues $0 \leq \lambda_1, \lambda_2, \dots, \lambda_n$,

this is the spectrum of graph $G(X)$, and corresponding eigenvectors $\phi_1, \phi_2, \dots, \phi_n$.

Let A be a $n \times n$ square matrix whose columns correspond to the eigenvectors $\phi_1, \phi_2, \dots, \phi_n$. And, let Λ be the diagonal matrix with diagonal elements $\lambda_1, \lambda_2, \dots, \lambda_n$. A can be factored as,

$$A = \Phi \times \Lambda \times \Phi^{-1}$$

7. Output results to an Envi image For each pixel

$$[i, j, x]$$

where (i, j) are the location coordinates, and $x = \phi_x$ is the eigenvector corresponding to that pixel in the following way, for each element $A_{i,j}$ of matrix A ,

$$A_{i,j} \rightarrow \left[\frac{i - i \pmod{m}}{m}, i \pmod{m}, j \right]$$

Where $\left[\frac{i - i \pmod{m}}{m}, i \pmod{m} \right]$ are the location coordinates of the pixel in the image, and j is the eigenvector.

7.4 Output

If the graph G generated by Laplacian Eigenmaps from image P is connected, $\lambda_1 = 0$ (smallest eigenvalue), and when it is not $\lambda_1 > 0$. The closer λ_1 is to zero, the stronger connected is the big component of the underlying graph. The first (smallest) eigenvalues and eigenvectors aid in clustering and classification of the data, specially λ_1 and associated eigenvector ϕ_1 are linked to the main clusters of the data, and are also associated with the optimal cut of the graph, also identified as the optimal cluster[13]. The second eigenvalue λ_2 quantifies how well connected G is [12].

The most relevant information about the structure and connections of the underlying graph is provided by the first or leading eigenvectors and corresponding eigenvalues. Since these eigenvectors are more resistant to normal fluctuations in the data (such as shades and small changes in the coloration of the same component), yet they reflect minimal changes that are important when the aim is to separate the graph into different components [10]. On the other hand, eigenvalues are affected by all changes, including those that are irrelevant [10]. For this reason, and given that our objective is to find anomalous pixels;

we use the information contained in the eigenvectors of G , and the corresponding eigenvalues for indexing in ascending order.

The output of our Laplacian Eigenmaps algorithm for image P with n pixels, is an Envi image of the same size on n bands that represent the eigenvectors $\phi_1, \phi_2, \dots, \phi_n$ corresponding to the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ ordered from smallest to largest. In other words, the spectrum for each pixel is the eigenvector corresponding to that pixel.

Pixels have different shades of gray on the single band Envi grayscale projections of the results of Laplacian Eigenmaps, together with the spectrum of many spectral bands. The values of each spectrum on the output of the program are the components of the eigenvectors in the n -dimensional space.

7.5 Social Network with Anomaly example

We can construct a random social network on n vertices with one anomalous vertex, and apply Laplacian Eigenmaps algorithm.

We create a simple random network R with 100 people, $n = 100$, randomly connected and let one vertex be an anomalous person.

Let R be a simple graph with vertex set

$$V(R) = [0, 1, 2, \dots, 99]$$

The vertices represent the people in our social network.

Generate $2 \times n$ edges and connect vertices at random. Resulting in edge set $E(R)$ that represents the connections among the people in the network.

To create an anomaly we first need to define what an anomaly is in our social network. An anomaly is an individual that has an abnormal number of connections to other individuals in the network. We take the last vertex and randomly connect it to $n/2$ vertices in R . The anomaly in this case is an individual that is substantially more connected than the rest of the vertices.

7.5.1 Results

We find the Laplacian of the graph R as defined in Section 7.1. Following the algorithm in Section 7.3 from step 4 through step 6, we obtained the following results 7.1, refer to the appendix PRO GraphSpectracircle for code.

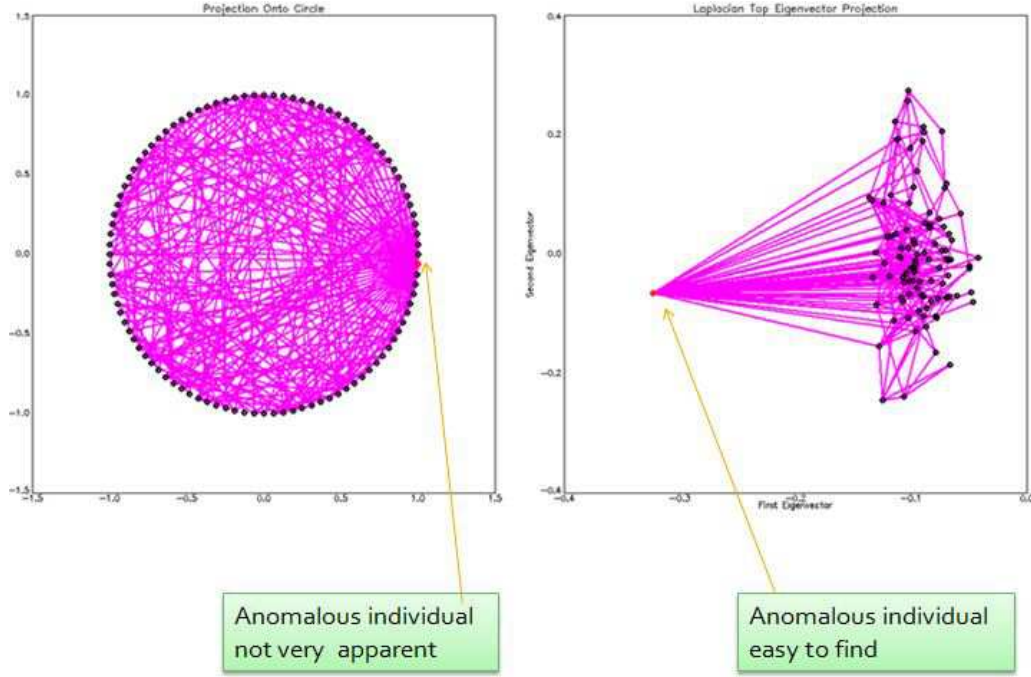


Figure 7.1: Social Network with anomaly results

The figure on the left shows the points forming a circle, with the anomalous individual shown in red. The figure on the right is the projection of the individuals onto the plane formed by two eigenvectors, in this case ϕ_1 and ϕ_2 , the x axis ranges from $\min[\phi_1]$ to $\max[\phi_1]$, and the y axis ranges from $\min[\phi_2]$ to $\max[\phi_2]$. Each node corresponds to an individual, placed according to the

coordinates of the eigenvector projected on the 2-dimensional space formed by ϕ_1 and ϕ_2 , nodes for individuals i and j are connected if their corresponding entry in the adjacency matrix W is 1.

$$A(i, j) = 1$$

The anomalous individual sticks out from the rest as expected, having a significantly larger number of links to other vertices, and taking values for the first two eigenvectors that are distant from the rest of the pixels. Whereas the majority of the pixels are located close to each other in the projection onto the first two eigenvectors.

Chapter 8

Laplacian Eigenmaps for Anomaly Detection in Spectral Images Experiments

For our Laplacian Eigenmaps experiments, we generated several images with known anomalies. To produce the test images with anomalies, we took some chips of different sizes from the WorldView-2 image 6.2, and replaced selected pixels on the images with anomalous pixels. An anomalous pixel is defined as pixel that comes from a material that does not belong in the test image, and therefore has a very distinct spectral profile.

We ran the program for the test images and the test images with anomalies. The threshold t is set, so after calculating all the Euclidean distances from pixel i to pixel j in the image, and ranking the distances from smallest to largest, our algorithm uses the distances that are equal or smaller than the threshold. This means that many times when the spectrum of anomalous pixels is significantly

different from the spectra of the rest of the pixels in the image (Euclidean distance from pixel i to pixel $j \geq t$) no edges are drawn. For this reason sometimes we obtain isolated vertices in the graph representation of images, when their spectra is very disparate, and correspond to anomalous pixels. If an anomalous object is large and comprises a few pixels in the image, or when working with larger images it is possible to obtain a separate small component on the graph that is highly connected within itself.

8.1 Example: Simple Chip, Simple Chip with Anomaly

- **Simple chip:**

Is a 10x10 chip of vegetation that is part of WorldView-2 image 6.2 shown in 8 bands, and that exhibits a similar spectral profile for all its pixels.

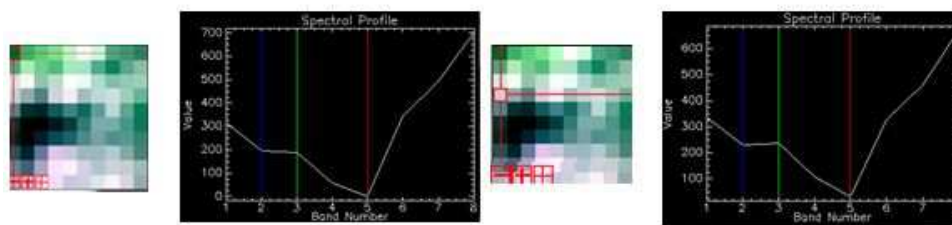


Figure 8.1: Simple Chip and spectral profile for two pixels

Image 8.1 shows Simple Chip and the spectra of two pixels, note that the pixels have very similar spectral profiles.

- **Simple Chip with Anomaly:**

To add an anomaly we started with the Simple Chip of vegetation 8.1, and replaced the top right corner pixel of the image with a pixel from a different image with a very different spectral profile 8.2.

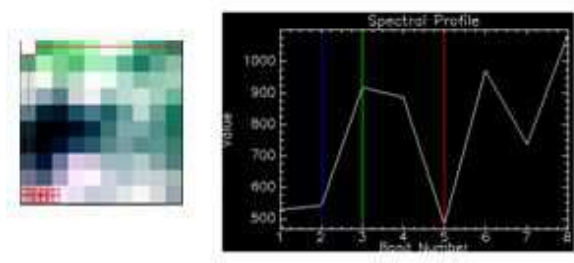


Figure 8.2: Simple Chip with Anomaly and spectral profile for anomalous pixel

Image 8.2 shows Simple Chip with an Anomaly and the spectral profile of the anomalous pixel, observe how the spectral profile is very different, with less blue (band 2), and more than double green (band 3), compared to the spectral profile of the background pixels in Figure 8.1.

8.1.1 Results

For this example, the threshold t is set at 50 percent.

Graph 8.3 shows a plot of the spectrum (eigenvalues) of Simple Chip (in blue) versus the spectrum of Simple Chip with Anomaly (in red), there is not much variation on most of the eigenvalues, the major difference is that Simple Chip with Anomaly has one eigenvalue equal to 1.00, and Simple Chip does not.

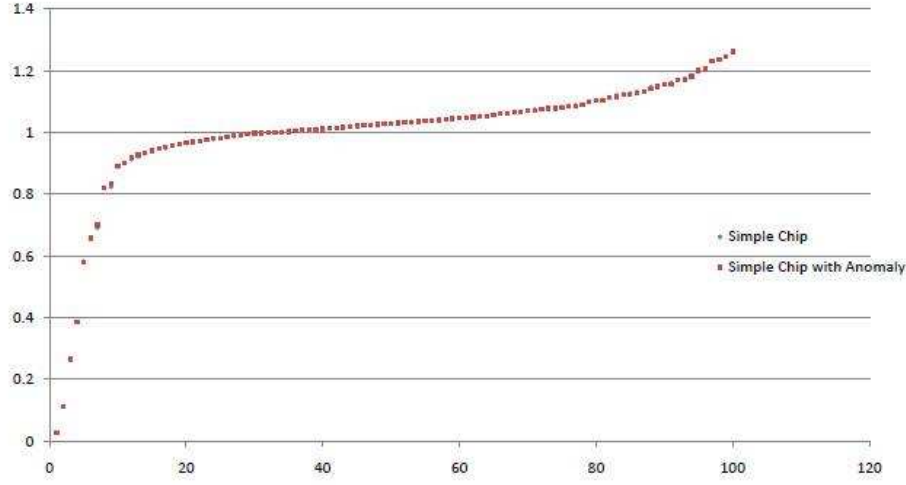


Figure 8.3: Spectrum of the graph for the Simple Chip 8.1, and Simple Chip with Anomaly 8.2

Figure 8.4 shows the projections for the output of the program for Simple Chip. Left, spectra of the pixels on spectral bands 1(blue) and 4 (red) with pixels i and j connected if their corresponding entry in the adjacency matrix A is 1. The projection of the spectra of the pixels on spectral bands 1 and 4 for Simple Chip shows all the points located close together and connected. Right, the projection of the pixels onto the plane formed by the first two eigenvectors ϕ_1 and ϕ_2 (IDL starts counting from 0, that is why in the graph we see eigenvectors 0 and 1. In reality we are referring to the first two eigenvectors ϕ_1 and ϕ_2). The x axis ranges from $\min[\phi_1]$ to $\max[\phi_1]$, and the y axis ranges from $\min[\phi_2]$

to $\max[\phi_2]$. Each node corresponds to a pixel, placed according to the coordinates of the eigenvector projected on the 2-dimensional space formed by ϕ_1 and ϕ_2 , with pixels i and j connected if their corresponding entry in the adjacency matrix W is 1.

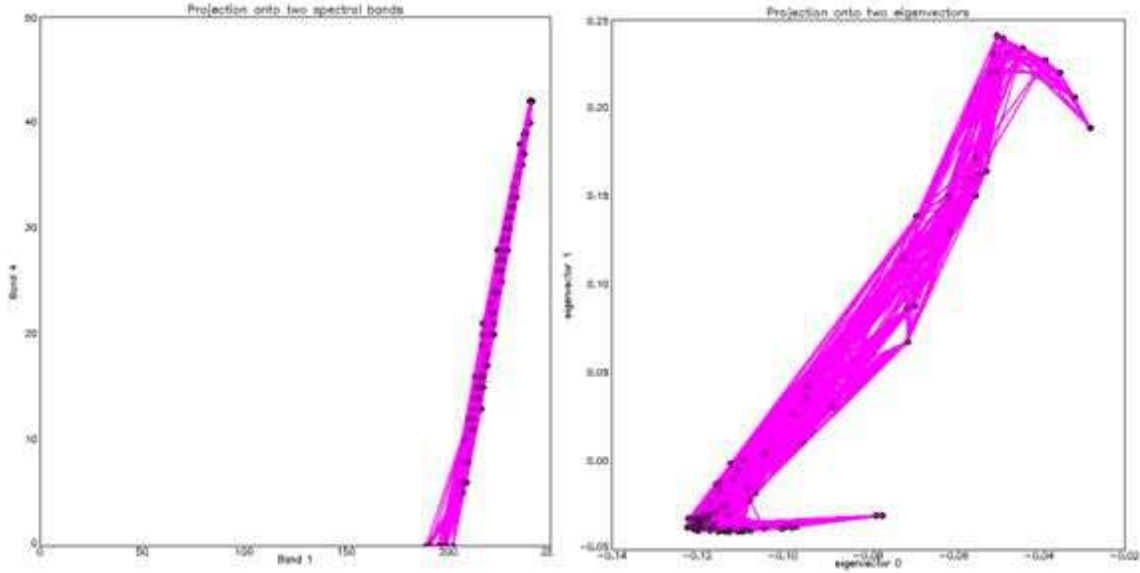


Figure 8.4: Left, projection of the spectra of the pixels on spectral bands 1 (blue) and 4 (red). Right, projection onto eigenvectors 0 and 1 of the Laplacian for Simple Chip 8.1

Similarly, Figure 8.5 shows the projections for the output of the algorithm for Simple Chip. The projection the spectra of the pixels on spectral bands 1 and 4 for Simple Chip with Anomaly shows the anomalous vertex is isolated and far from the rest of the vertices that are close together and connected. Right, the projection of the pixels onto the plane formed by first two eigenvectors ϕ_1

and ϕ_2 , observe how adding a single anomaly alters the projections from Simple Chip 8.4 to Simple Chip with Anomaly 8.5.

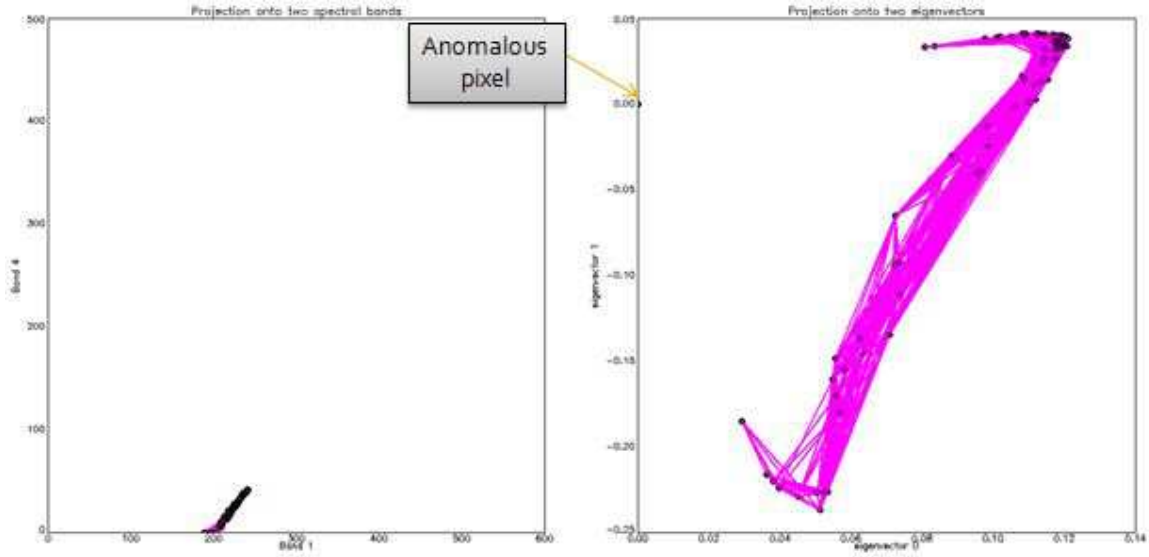


Figure 8.5: Left, projection of the spectra of the pixels on spectral bands 1 (blue) and 4 (red). Right, projection onto eigenvectors 0 and 1 of the Laplacian for Simple Chip with Anomaly 8.2

The grayscale projection results for Simple Chip eigenvector 1 are shown on Figure 8.6 on the images to the left, the Eigenvector Profiles for two pixels are shown to the right.

The grayscale projection results for Simple Chip with Anomaly eigenvector 1 are shown on Figure 8.7 on the images to the left, the anomalous pixel shows in white, the Eigenvector Profiles for the anomaly and the pixel next to it are shown to the right, the anomaly has a zero value for all bands except for a spike to 1 in band 33, in comparison, the other pixel takes on values different that zero

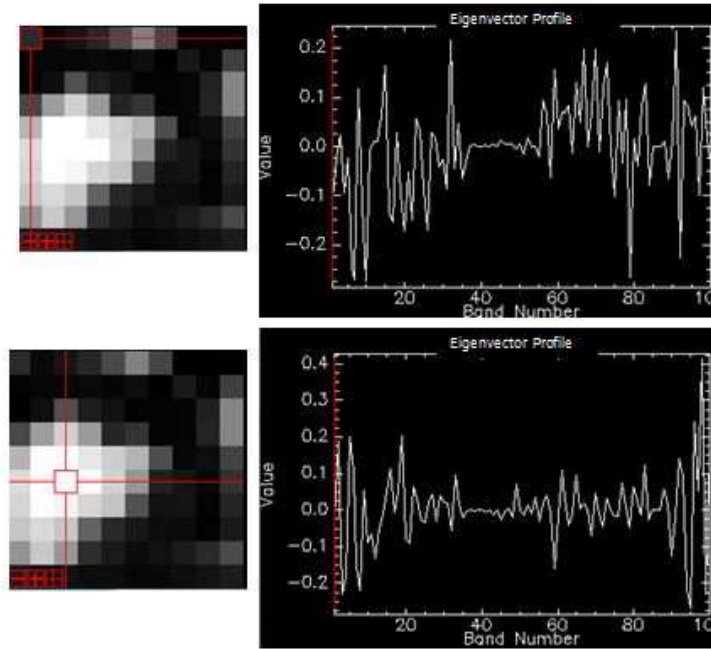


Figure 8.6: Left, grayscale projection results for Simple Chip eigenvector 1. Right, Eigenvector Profiles for two pixels

for all bands.

For this example, Laplacian Eigenmaps shows promising results, it helps to identify the anomalous pixel. It is interesting that the Eigenvector Profile for the background pixels takes on values different than zero for most bands (eigenvectors) and oscillating within the values 0.3 to -0.3, whereas the Eigenvector Profile for the anomalous has a zero value for all bands except for a single spike to 1. This is consistent with the idea mentioned earlier about eigenvectors being indicator vectors, the anomaly is in one eigenvector only, and is not part of the main component obtained.

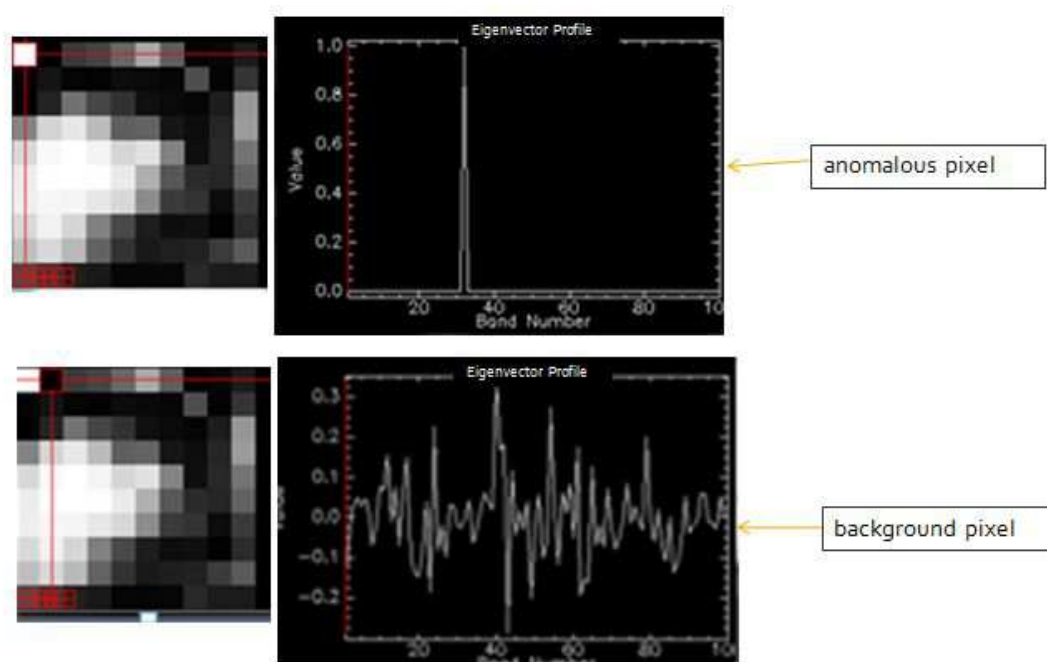


Figure 8.7: Left, grayscale projection results for Simple Chip with Anomaly eigenvector 1. Right, Eigenvector Profiles for the anomaly and the pixel next to it

8.2 Simple Chip with 3 Anomalies

For this experiment, we started with Simple Chip of vegetation 8.1 used in the previous section, and replaced three pixels in different parts of the image with pixels from a different image with a very different spectral profile.

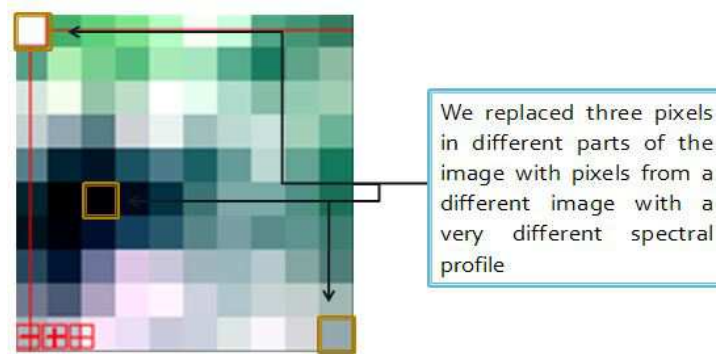


Figure 8.8: Simple Chip with 3 Anomalies

8.2.1 Results

For this example, the threshold t is set at 50 percent.

Graph 8.9 shows a plot of the spectrum of Simple Chip with 1 Anomaly 8.2 (in blue) vs. Spectrum of Simple Chip with 3 Anomalies (in red). There is not much variation between the two plots, except that there are three eigenvalues equal to 1.

Figure 8.10 shows the projections for the output of the algorithm for Simple

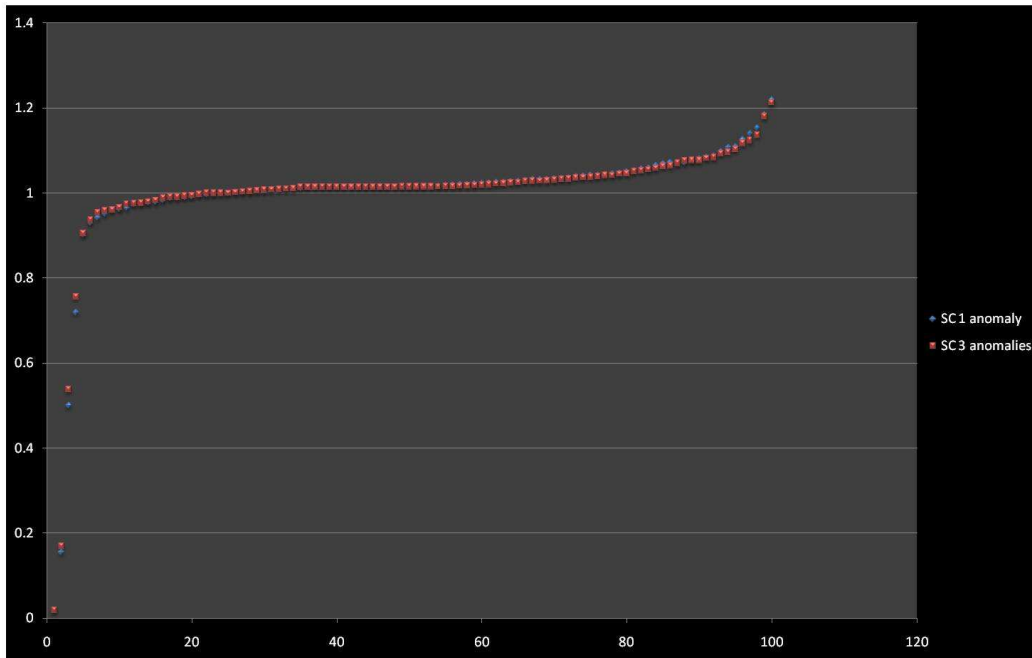


Figure 8.9: Spectrum of Simple Chip with 1 anomaly vs. Spectrum of Simple Chip with 3 Anomalies

Chip with 3 Anomalies. The spectra of the pixels on spectral bands 1 and 4 (left), and the projection onto the plane formed by the first two eigenvectors, the anomalous pixels are stacked together on 0 for both eigenvectors.

The grayscale projection results for Simple Chip with 3 Anomalies eigenvector 1 are shown on Figure 8.11 on the image to the left, the Eigenvector Profile for a background pixel is shown to the right.

The grayscale projection results for Simple Chip with 3 Anomalies eigenvector 1 are shown on Figure 8.12 on the images to the left, the anomalous

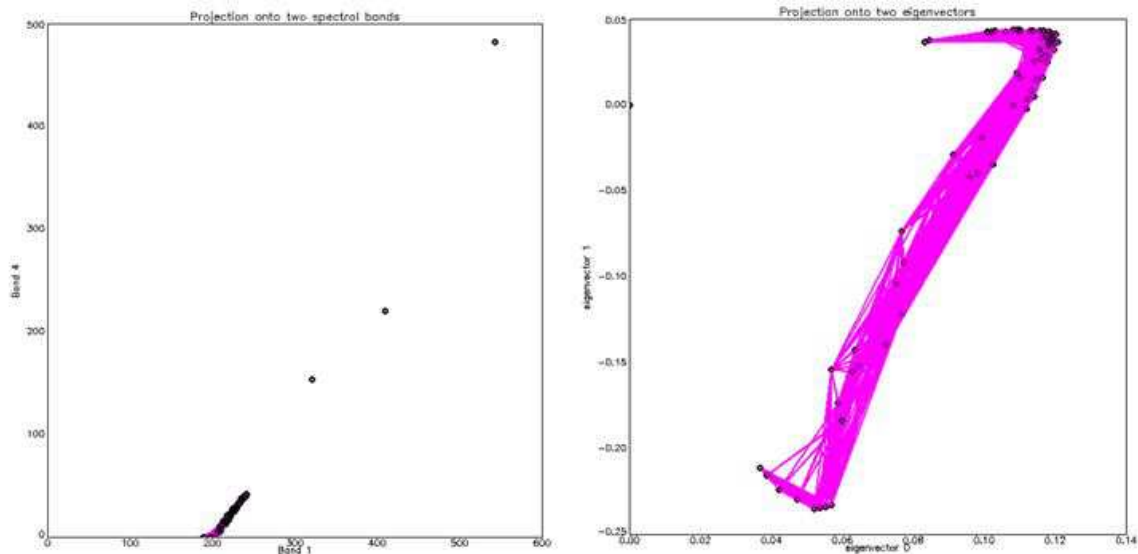


Figure 8.10: Left, projection of the spectra of the pixels on spectral bands 1 (blue) and 4 (red). Right, projection onto eigenvectors 0 and 1 of the Laplacian for Simple Chip with 3 Anomalies 8.8

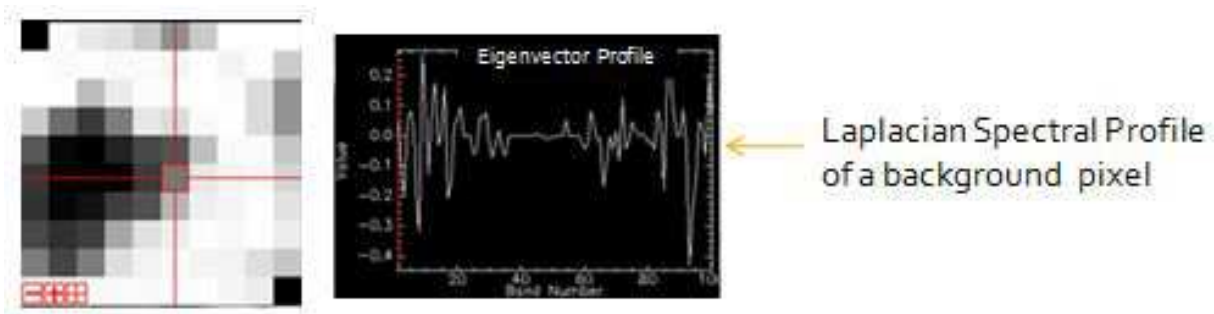


Figure 8.11: Left, grayscale projection results for Simple Chip with 3 Anomalies eigenvector 1. Right, Eigenvector Profile for background pixel

pixels are highlighted, and their Eigenvector Profiles are shown to the left of every image. Again we see the same behavior as Simple Chip with Anomaly, the Eigenvector Profiles for the anomalies have a zero value for all bands except for a spike to 1 in band 23 for the first anomaly(top left), a spike to -1 in band

22 for the second anomaly (top right), and a spike to 1 in band 24 for the third anomaly (bottom left). These spikes might be useful in identifying anomalous pixels, anomalies seem consistent here.

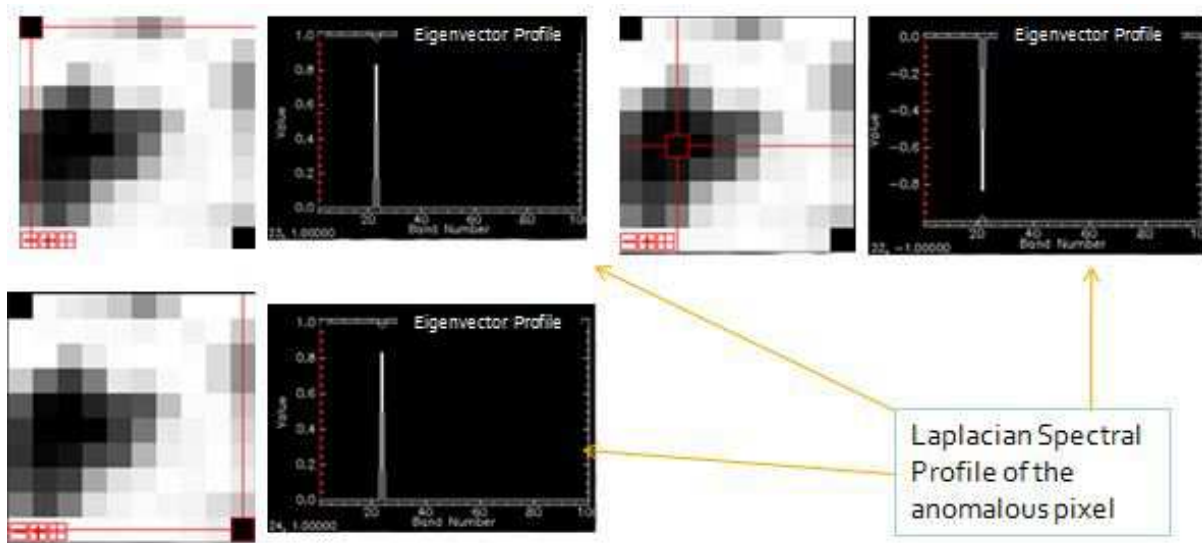


Figure 8.12: Left, grayscale projection results for Simple Chip with Anomaly eigenvector 1. Right, Eigenvector Profiles for the anomaly and the pixel next to it

8.3 Laplacian Anomaly Image

The Laplacian Anomaly Image is a 50x50 complex chip that is part of WorldView-2 image 6.2. It contains an assortment of materials with different spectral composition, such as some vegetation, the ceiling of a building, a road, three cars, among others. Shown on 8.14.



Figure 8.13: WorldView-2 image, Laplacian Anomaly Image 50x50 Chip location shown on red box



Figure 8.14: Laplacian Anomaly Image- 50x50 Chip

8.3.1 Results

The threshold t is set at 30 percent for this experiment.

Figure 8.15 shows the projections for the output of the algorithm for Laplacian Anomaly Image. To the left, the projection of the spectra of the pixels on spectral bands 1 and 4. Right, the projection of the pixels onto the plane formed by the first two eigenvectors ϕ_1 and ϕ_2 .

Figure 8.16 shows the grayscale projection results for Laplacian Anomaly Image eigenvector 1 (left), and Eigenvector Profiles for three background pixels (right). The Eigenvector Profile for these three pixels of different components

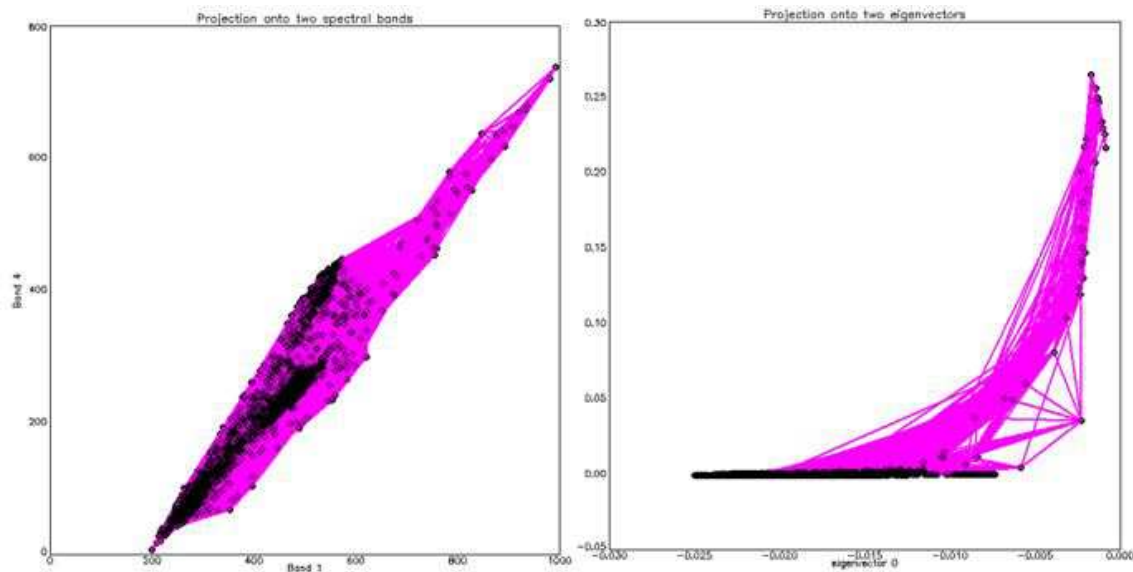


Figure 8.15: Left, projection of the spectra of the pixels on spectral bands 1 (blue) and 4 (red). Right, projection onto eigenvectors 0 and 1 of the Laplacian for Laplacian Anomaly Image 8.14

of the image shows a lot of variation mainly within values 0.05 and -0.05, and has values different than zero for almost every single band.

In comparison, figure 8.17 shows the grayscale projection results for Laplacian Anomaly Image eigenvector 1 (left), and Eigenvector Profiles for the anomalous or different pixels (right). The spikes or maximum values for the Eigenvector Profile of anomalies are larger than the values for the background pixels. The Eigenvector Profiles for those pixels that stick out and that seem to be cars parked on the road outside the building, have a zero value for the majority bands, with some spikes to 0.4 and -0.6.

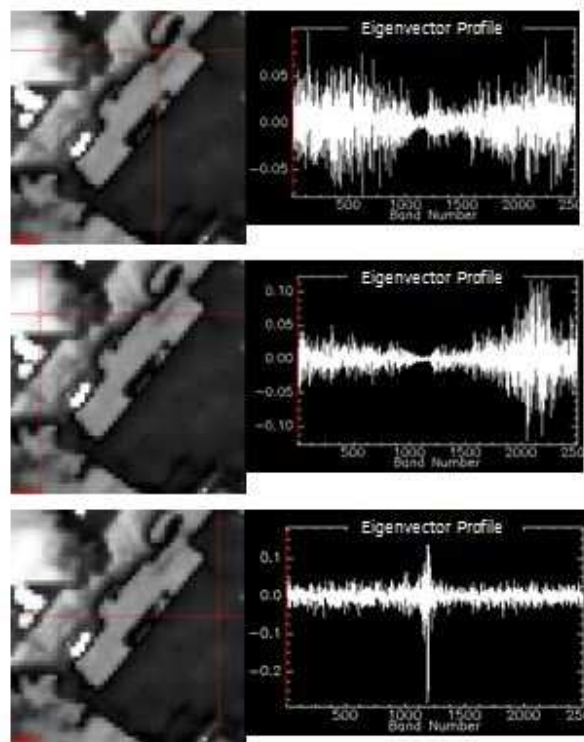


Figure 8.16: Left, grayscale projection results for Laplacian Anomaly Image eigenvector 1. Right, Eigenvector Profile for 3 background pixels

The results of the program visualized in three dimensions for bands (eigenvectors) 1, 2, 3 are shown in 8.18 (left), most of the points are close together with the exception of a few points that are far away from the rest, we selected these points and created a new class that is shown in red. The image on the right, shows the grayscale projection for eigenvector 1, the pixels in red correspond to the selected class, and the most dissimilar pixels in the 8.14 image.

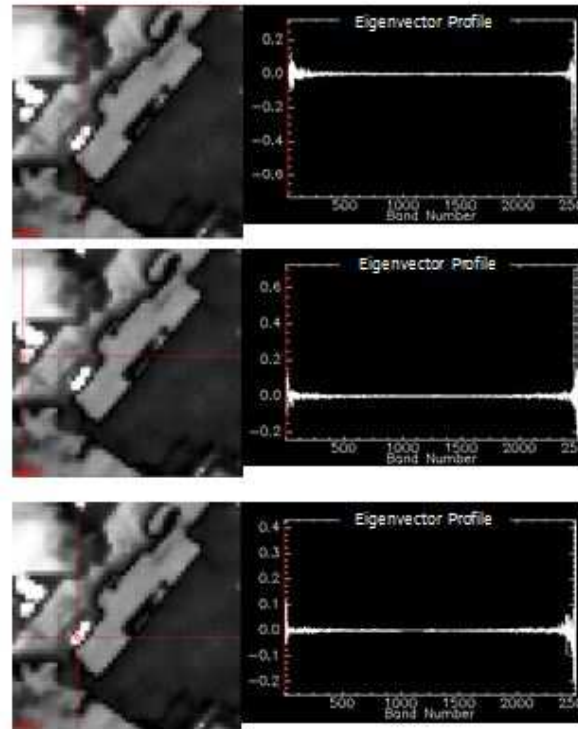
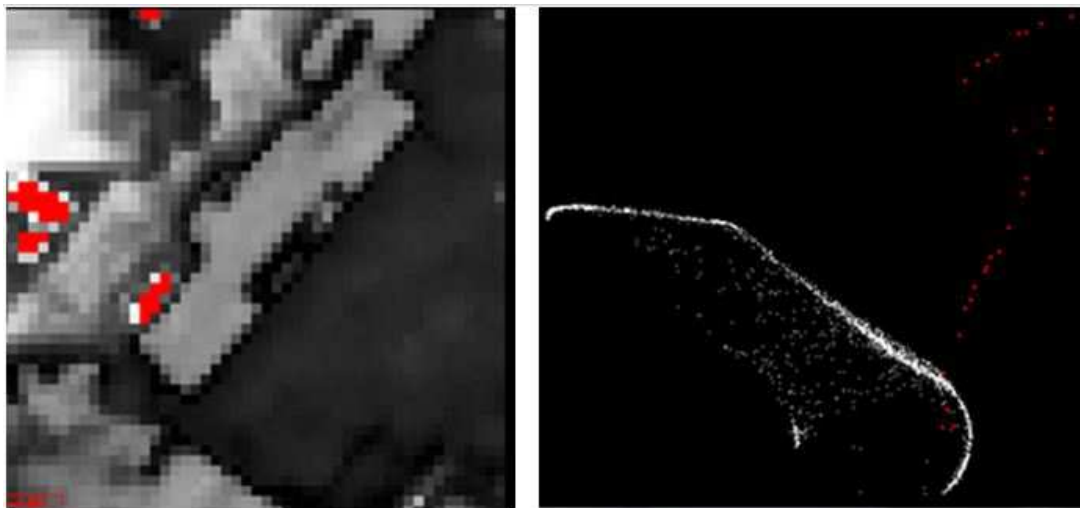


Figure 8.17: Left, grayscale projection results for Laplacian Anomaly Image eigenvector 1. Right, Eigenvector Profiles for the anomalous (different) pixels

The Laplacian Eigenmaps program successfully identified the anomalous pixels in the image, as pixels that are far apart from the rest of the pixels, and that form a small cluster of their own.



n-d visualizer for bands 1,2, and 3

Figure 8.18: n-d visualizer for bands 1,2,3, and a selected class (red) that includes the most distant points (left). Projection of the results on Gray Scale results for band 2(right)

Chapter 9

Conclusions

The Laplacian Eigenmaps program successfully identified anomalous pixels in the experiments performed. By using the information provided by the first eigenvectors of the Laplacian matrix of the graph constructed from the image, we were able to find information that could not be easily visualized with the original data provided by the spectral image.

The anomalies are visually apparent without a great amount of effort on the Envi projections of the results of the program. And their Eigenvector Profiles consistently show a similar behavior with zero for most bands and big spikes in the bands where the anomalous spectrum is present. It is still a question why these spikes on the spectra of the anomalies happen, but the anomalies seem show this behavior time and again. We believe it is related to the idea of eigenvectors being indicator vectors, and point out anomalies that are not present in the major clusters of the data, but form highly connected clusters of

their own.

Chapter 10

Future Work

At the moment we are working with 50x50 chips of big images, we would like to create a procedure to divide the image into multiple tiles and expand our program to iteratively separate major components of the graph from different or anomalous components . The next step would be to make Laplacian Eigenmaps into an anomaly detection algorithm.

The Laplacian Eigenmaps algorithm provides an enormous amount of information about the graph used to represent the data that could have other uses in imaging processing, and not just related to anomaly detection.

Our current algorithm creates a simple graph with no weighted edges, it would be interesting to assign weights to edges according to how similar the spectra of the pixels is. This will give us more insight on the structure of the graph; since it would exploit the heat kernel and aid to find a geometric characterization of the graph, by computing the geodesic and Euclidean distances of

the vertices in the manifold vector space, i.e. shortest path between each pair of vertices in the manifold, which can be used to determine and encapsulate the structure of the manifold M . This could be useful for classification and clustering of the data[14].

Appendix A

PRO GraphSpectracircle

```

sample_graph=2
if sample_graph eq 2 then begin ; random graph with anomaly
    seed=systime(1)
    n_vertices = 100;list of vertices
    V = indgen(n_vertices)
    deg = intarr(n_vertices)
    print, 'vertices: '
    print, V
    n_edges=n_vertices*2; list of edges
    E = floor((n_vertices-1)*(randomn(seed,2,n_edges,uniform=1)))
    print, 'E:'
    print, E
    for i=0,(n_vertices/2-2) do begin
        E = transpose([transpose(E),transpose([n_vertices-1,i*2])])
    endfor
    edge_list = E
    size_E = size(E)
    n_edges = size_E[2]
    print, 'edges: '
    print, E
endif

;Compute degrees of edges and create matrices L and T
;Actually, we comute Tsqrt=T^(-1/2) which is used to compute
;the Laplacian. Also, we compute L and T at the same to
;avoid looping though vertices multiple times.

```

```

L = intarr(n_vertices,n_vertices)
Tsqr = fltarr(n_vertices,n_vertices)
for i=0,n_edges-1 do begin
  if (E[0,i] NE E[1,i]) then begin
    L[E[0,i],E[1,i]] = -1
    L[E[1,i],E[0,i]] = -1
  endif
  deg(E[0,i]) = deg(E[0,i])+1
  deg(E[1,i]) = deg(E[1,i])+1
endfor
for i=0,n_vertices-1 do begin
  L[i,i]=deg[i]
  Tsqr[i,i]= 1/sqrt(deg[i])
endfor
print, 'Graph L:'
print, L
print, 'Graph T-1/2:'
print, Tsqr

; Compute the Laplacian
Lap=Tsqr#L#Tsqr
print, 'Laplacian:'
print, Lap

; Compute the eigenvalues and eigenvectors of the Laplacian
A = Lap
TRIRED, A, D, E
; Compute the eigenvalues (returned in vector D) and
; the eigenvectors (returned in the rows of the array A):
TRIQL, D, E, A
; Print eigenvalues and eigenvectors

;print, '
;print, 'Eigenvalues: '
;print, D
;print, ' '
;print, 'Eigenvectors: (rows of the following array) '
;print, A

```

```

;print, A[:,0]

if sample_graph eq 3 then begin; for random graph with one anomaly
  !P.MULTI = [0, 1,3]
  window, 0, xsize=400, ysize=600
  D_idx = sort(D)
  x_range=[min(A[:,D_idx[0]]),max(A[:,D_idx[0]])]
  x_range=[x_range[0]-(x_range[1]-x_range[0])/10,x_range[1]+
    (x_range[1]-x_range[0])/10]
  y_range=[min(A[:,D_idx[1]]),max(A[:,D_idx[1]])]
  y_range=[y_range[0]-(y_range[1]-y_range[0])/10,y_range[1]+
    (y_range[1]-y_range[0])/10]
  plot, A[0:1,D_idx[0]], A[0:1,D_idx[1]], psym=4, symsize=0.5, $
    xrange=x_range, yrange=y_range, background=!P.COLOR,
  color=0, $; set up color and title for plot
    thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
    title='Laplacian Top Eigenvector Projection', $
    xtitle='First Eigenvector', $
    ytitle='Second Eigenvector'
  for edge=0,n_edges-1 do begin
    nodes = [edge_list[0,edge],edge_list[1,edge]]
    print, nodes
    oplot, A[nodes,D_idx[0]], A[nodes,D_idx[1]], $
      psym=0, symsize=0.5, $
      thick=2, color='FF00FF'x
  endfor
  oplot, A[(n_vertices-2):(n_vertices-1),D_idx[0]], A[(n_vertices-2):
    (n_vertices-1),D_idx[1]], psym=4, symsize=0.5, $
    thick=2, color='0000FF'x
  oplot, A[0:(n_vertices-2),D_idx[0]], A[0:(n_vertices-2),D_idx[1]],
  psym=4, symsize=0.5, $
    thick=2, color=0

  plot, A[:,D_idx[0]], $
    background=!P.COLOR, color=0, $
    thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
    title='First Eigenvector'
  plot, A[:,D_idx[1]], $

```

```

        background=!P.COLOR, color=0, $
        thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
        title='Second Eigenvector'

window, 1, xsize=400, ysize=600
e1 = n_vertices-2
e2 = n_vertices-3
D_idx = sort(D)
x_range=[min(A[:,D_idx[0]]),max(A[:,D_idx[0]])]
x_range=[x_range[0]-(x_range[1]-x_range[0])/10,x_range[1]+
(x_range[1]-x_range[0])/10]
y_range=[min(A[:,D_idx[1]]),max(A[:,D_idx[1]])]
y_range=[y_range[0]-(y_range[1]-y_range[0])/10,y_range[1]+
(y_range[1]-y_range[0])/10]
plot, A[0:1,D_idx[e1]], A[0:1,D_idx[e2]], psym=4, symsize=0.5, $
        xrange=x_range, yrange=y_range, background=!P.COLOR,
color=0, $; set up color and title for plot
        thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
        title='Laplacian Top Eigenvector Projection', $
        xtitle='First Eigenvector', $
        ytitle='Second Eigenvector'
for edge=0,n_edges-1 do begin
    nodes = [edge_list[0,edge],edge_list[1,edge]]
    print, nodes
    oplot, A[nodes,D_idx[e1]], A[nodes,D_idx[e2]], $
        psym=0, symsize=0.5, $
        thick=2, color='FF00FF'x
endfor
oplot, A[(n_vertices-2):(n_vertices-1),D_idx[e1]], A[(n_vertices-2):
(n_vertices-1),D_idx[e2]], psym=4, symsize=0.5, $
        thick=2, color='0000FF'x
oplot, A[0:(n_vertices-2),D_idx[e1]], A[0:(n_vertices-2),
D_idx[e2]], psym=4, symsize=0.5, $
        thick=2, color=0
plot, A[:,D_idx[e1]], $
        background=!P.COLOR, color=0, $
        thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$

```



```

        title='First Eigenvector'
    plot, A[*,D_idx[e2]], $
        background=!P.COLOR, color=0, $
        thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
        title='Second Eigenvector'
endif

if sample_graph eq 2 then begin; for random graph with one anomaly
    !P.MULTI = [0, 2, 1]
    window, 0, xsize=1200, ysize=600

    D_idx = sort(D)
    x_range=[-1.2,1.2]
    y_range=[-1.2,1.2]
    x_coords=cos(indgen(n_vertices)*2*3.141597/n_vertices)
    y_coords=sin(indgen(n_vertices)*2*3.141597/n_vertices)
    plot, x_coords[0:1], y_coords[0:1], psym=4, symsize=0.5, $
        xrange=x_range, yrange=y_range, background='FFFFFF'x,
    color=0, $; set up color and title for plot
        thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
        title='Projection Onto Circle'
    for edge=0,n_edges-1 do begin
        nodes = [edge_list[0,edge],edge_list[1,edge]]
        print, nodes
        oplot, x_coords[nodes], y_coords[nodes], $
            psym=0, symsize=0.5, $
            thick=2, color='FF00FF'x
    endfor
    oplot, x_coords[(n_vertices-2):(n_vertices-1)], y_coords[(n_vertices-2):
(n_vertices-1)], psym=4, symsize=0.5, $
        thick=2, color='0000FF'x
    oplot, x_coords[0:(n_vertices-2)], y_coords[0:(n_vertices-2)],
    psym=4, symsize=0.5, $
        thick=2, color=0

    D_idx = sort(D)
    x_range=[min(A[*,D_idx[0]]),max(A[*,D_idx[0]])]

```

```

x_range=[x_range[0]-(x_range[1]-x_range[0])/10,x_range[1]+
(x_range[1]-x_range[0])/10]
y_range=[min(A[:,D_idx[1]]),max(A[:,D_idx[1]])]
y_range=[y_range[0]-(y_range[1]-y_range[0])/10,y_range[1]+
(y_range[1]-y_range[0])/10]
plot, A[0:1,D_idx[0]], A[0:1,D_idx[1]], psym=4, symsize=0.5, $
    xrange=x_range, yrange=y_range, background=!P.COLOR,
    color=0, $; set up color and title for plot
    thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
    title='Laplacian Top Eigenvector Projection', $
    xtitle='First Eigenvector', $
    ytitle='Second Eigenvector'
for edge=0,n_edges-1 do begin
    nodes = [edge_list[0,edge],edge_list[1,edge]]
    print, nodes
    oplot, A[nodes,D_idx[0]], A[nodes,D_idx[1]], $
        psym=0, symsize=0.5, $
        thick=2, color='FF00FF'x
endfor
oplot, A[(n_vertices-2):(n_vertices-1),D_idx[0]], A[(n_vertices-2):
(n_vertices-1),D_idx[1]], psym=4, symsize=0.5, $
    thick=2, color='0000FF'x
oplot, A[0:(n_vertices-2),D_idx[0]], A[0:(n_vertices-2),
D_idx[1]], psym=4, symsize=0.5, $
    thick=2, color=0

endif

end

```

Appendix A

laplacianprojection.pro

```

pro Laplacian_Projection_define_buttons, buttonInfo
compile_opt STRICTARR

;compile_opt idl2
envi_define_menu_button, buttonInfo,
value='Laplacian Projection', $
    position='first', ref_value='User Functions',
uvalue='none', $
    event_pro='Laplacian_Projection'
end

PRO Laplacian_Projection_doit, fid, dims, plot_distances,
plot_eigenvalues

; read in the image
ENVI_FILE_QUERY, fid, fname=fname, nb=nb
; get information from image on the number of bands
rows = dims[4]-dims[3]+1
cols = dims[2]-dims[1]+1
bands = nb
Im = fltarr(cols,rows,bands)
; sets up array to hold the image
for band=0,nb-1 do begin
    Im[:,*,band] = ENVI_GET_DATA(fid=fid, dims=dims,
    pos=band)
endfor

```

```

!P.MULTI = 0 ; 1 plot per window

; X is an array that has the spectra of the pixels
as columns
; So the spectra of pixel i is X[i,*]
X = double(reform(Im, rows*cols,bands))

; List of vertices and vertex information
n_vertices = rows*cols
V = indgen(n_vertices)
deg = intarr(n_vertices)

;print, 'vertices: '
;print, V

; Compute the distance matrix D.
; D[i,j] is the distance from the spectra of pixel i
to the spectra of pixel j
Dist = fltarr(n_vertices,n_vertices)
for i=0,n_vertices-1 do begin
    for j=0,n_vertices-1 do begin
        Dist[i,j]=norm(X[i,*]-X[j,*])
    endfor
endfor

; Sort the distances to find a good threshold
threshold_percent= 0.3
S = sort(Dist)
Dist_list = Dist[S]
threshold = Dist_list[floor(n_elements(Dist_list)*
threshold_percent)]
threshold_line = fltarr(n_elements(Dist_list))
threshold_line[*] = threshold

; Create the adjacency matrix from the threshold.
Id = fltarr(n_vertices,n_vertices)

```

```

for i=0,n_vertices-1 do begin
    Id(i,i) = 1
endfor
Adj = (Dist LE threshold)

; Plot the distances.
if (plot_distances eq 1) then begin
    window, 0
    plot, Dist_list, background=!P.COLOR, color=0
    oplot, threshold_line, color='0000A0'x
endif

; Compute the degrees of the vertices.
deg = fix(total(Adj,1))
;print, 'deg'
;print, deg

; Compute degrees of edges and create matrices L and T
; Actually, we compute Tsqrt=T^(-1/2) which is used to compute
; the Laplacian. Also, we compute L and T at the same to
; avoid looping though vertices multiple times.
L = intarr(n_vertices,n_vertices)
Tsqrt = fltarr(n_vertices,n_vertices)
I= Identity(n_vertices)
L = I-Adj
;print, 'I'
;print, I
;print, 'Adj'
;print, Adj
;print, 'L'
;print, L

for i=0,n_vertices-1 do begin
    L[i,i]=deg[i]
    Tsqrt[i,i]= -1/sqrt(deg[i])
endfor
;print, 'Graph L:'
;print, L

```

```

;print, 'Graph  $T^{(-1/2)}$ :'
;print, Tsqrt

; Compute the Laplacian
Lap=Tsqr#L#Tsqr
; print, 'Laplacian:'
; print, Lap

; Compute the eigenvalues and eigenvectors of the Laplacian
A = Lap
TRIRED, A, D, E

; Compute the eigenvalues (returned in vector D) and
; the eigenvectors (returned in the rows of the array A):
TRIQL, D, E, A
; Print eigenvalues and eigenvectors
;print, '
;print, 'Eigenvalues: '
;print, D
;print, ' '
;print, 'Eigenvectors: (rows of the following array) '
;print, A
;print, A[*,0]

;sort the eigenvectors and eigenvalues
eigen_order = sort(D)
D = D[eigen_order]
print, D
print, ' '

A_temp = A
for i = 0,n_vertices-1 do begin
A[*,i]=A_temp[*,eigen_order[i]]
endfor

plot_graph = 1
if plot_graph eq 1 then begin
    first_band = 1

```

```

second_band = 4
first_eigenvector = 1
second_eigenvector = 2
!P.MULTI = [0,2,1]
window, 1, xsize=1200, ysize=600

; plotting nodes on two bands of the image
x_range=[0,max(X[:,first_band])]
y_range=[0,max(X[:,second_band])]
plot, X[:,first_band], X[:,second_band], psym=4,
symsize=0.5, $
    xrange=x_range, yrange=y_range, background=!P.COLOR,
color=0, $

; set up color and title for plot
    thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
    title='Projection onto two spectral bands', $
    xtitle='Band '+strtrim(first_band,2), $
    ytitle='Band '+strtrim(second_band,2)
for i=0,n_vertices-1 do begin
    for j=0,n_vertices-1 do begin
        if (Adj[i,j] eq 1) then begin
            nodes = [i,j]
            oplot, X[nodes,first_band], X[nodes,second_band], $
                psym=0, symsize=0.5, $
                thick=2, color='FF00FF'x
        endif
    endfor
endfor
oplot, X[:,first_band], X[:,second_band], psym=4,
symsize=0.5, color=0

;plotting nodes on the Eigenvectors
D_idx = sort(D)
x_range=[min(A[:,D_idx[first_eigenvector]]),max(A[:,
D_idx[first_eigenvector]])]
y_range=[min(A[:,D_idx[second_eigenvector]]),max(A[:,
D_idx[second_eigenvector]])]

```

```

plot, A[:,D_idx[first_eigenvector]], A[:,D_idx[second_eigenvector]],
psym=4, symsize=0.5, $
    xrange=x_range, yrange=y_range, background=!P.COLOR, color=0,$;
    set up color and title for plot
    thick=2, ymargin=[5,5], charsize=0.5, ticklen=0,$
    title='Projection onto two eigenvectors', $
    xtitle='eigenvector '+strtrim(first_eigenvector,2), $
    ytitle='eigenvector '+strtrim(second_eigenvector,2)

for i=0,n_vertices-1 do begin
    for j=0,n_vertices-1 do begin
        if (Adj[i,j] eq 1) then begin
            nodes = [i,j]
            oplot, A[nodes,D_idx[first_eigenvector]], A[nodes,D_idx
            [second_eigenvector]], $
                psym=0, symsize=0.5, $
                thick=2, color='FF00FF'x
        endif
    endfor
endfor
oplot, A[:,D_idx[first_eigenvector]], A[:,D_idx[second_eigenvector]],
psym=4,
symsize=0.5, color=0
endif

; put image into memory
out_image = reform(A,cols,rows,rows*cols)
envi_enter_data, out_image

!P.MULTI = 0 ; 1 plot per window
end

pro Laplacian_Projection, ev
compile_opt STRICTARR
; compile_opt idl2

; Select input file and get relevant stats
envi_select, fid=fid, dims=dims, pos=pos, title=

```



```

'Select Input File for Laplacian Projection'
  if (fid[0] eq -1) then return
  base = widget_auto_base(title='Laplacian Projection
  Paramters')
  s1 = widget_base(base, /column, /frame)
  s2 = widget_base(s1, /row)
  param6 = widget_menu(s2, /auto, /exclusive,
prompt='Plot Distances: ', list=['No', 'Yes'], default_ptr=1,
uvalue='plot_distances')
  s2 = widget_base(s1, /row)
  param6 = widget_menu(s2, /auto, /exclusive, prompt=
'Plot Eigenvalues: ', list=['No', 'Yes'], default_ptr=1,
uvalue='plot_eigenvalues')
  res = auto_wid_mng(base)
  if (res.accept eq 0) then return
  plot_distances = res.plot_distances
  plot_eigenvalues = res.plot_eigenvalues

Laplacian_Projection_doit, fid, dims, plot_distances,
plot_eigenvalues

end

```

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